

Group structure of the integration-by-part identities and its application to the reduction of multiloop integrals

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ABSTRACT: The excessiveness of integration-by-part (IBP) identities is discussed. The Lie-algebraic structure of the IBP identities is used to reduce the number of the IBP equations to be considered. It is shown that Lorentz-invariance (LI) identities do not bring any information additional to that contained in the IBP identities, and therefore, can be discarded.

KEYWORDS: NLO Computations, QCD.

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1. Introduction

Calculation of the multiloop radiative corrections in different physical processes becomes more and more important nowadays. Mainly this is because of the increasing precision of the modern experiments, both in high-energy physics and in spectroscopy. The use of the IBP identities [1, 2] is a standard approach to the effective calculation of the loop integrals. These identities can reduce the problem of calculation of arbitrary integral with a given topology to that of calculation of the limited number of simpler integrals of the same topology and its subtopologies.

However, the application of the IBP identities is hampered by their infinite number. The problem is that it is not always clear which identities should be used to reduce a given integral. One standard approach, which has proved to be useful, is considering the identities starting from the simplest ones and creating a database of the rules for the reduction [3]. This algorithm is essentially sequential, since, in order to solve the next identity, it is necessary first to substitute all integrals which are already in the database. Another approach to the problem is to reduce it to the problem of “division with the remainder” with respect to some ideal [4]. The main difficulty on this way is to derive the Gröbner basis of the ideal allowing for the correct determination of the simplest remainder. According to ref. [5], this procedure essentially depends on the ordering chosen. This choice is, in some cases, not a simple problem, and the Gröbner basis then can be hardly found. Thus, the first approach (Laporta approach) appears to be necessary, at least, in this situation.

One of the problems which can spoil the effectiveness of the Laporta approach is the excessiveness of the IBP identities. When the IBP identities are considered one-by-one,

most identities do not give any additional information. Indeed, the number of the identities grows as the volume of the region in \mathbb{Z}^N multiplied by the number of identities in one point, $L(L + E)$ (L being the number of loops, E , the number of external momenta), while the number of integrals involved grows only as the volume of the region in \mathbb{Z}^N . Thus, in the asymptotics, only one identity out of $L(L + E)$ give new information. Other identities in the Laporta approach are checked to reduce to $0 = 0$. Unfortunately, this check can be a very time consuming calculation. The determination of the minimal set of the IBP identities is also important for the analytical solution of identities (i.e., the derivation of the reduction rules) as it allows one to consider a smaller set of the identities.

Though the algebraic manipulation with the IBP identities has been used for a long time, the observation that IBP identities form a closed Lie algebra has not been exploited so far. The main purpose of this paper is to demonstrate how the Lie-algebraic properties of the IBP equations can be used for both “division with the remainder” algorithms and Laporta-like algorithms.

2. General discussion

Assume that we are interested in the calculation of the L -loop integral depending on the E external momenta p_1, \dots, p_E . There are N scalar products depending on the loop momenta l_i :

$$\begin{aligned} s_{ik} &= l_i \cdot q_k, & 1 \leq i \leq L, k \leq L + E, \\ N &= L(L + 1)/2 + LE \end{aligned} \tag{2.1}$$

where $q_{1, \dots, L} = l_{1, \dots, L}$, $q_{L+1, \dots, L+E} = p_{1, \dots, E}$.

The loop integral has the form

$$J(\mathbf{n}) = J(n_1, n_2, \dots, n_N) = \int d^D l_1 \dots d^D l_L j(\mathbf{n}) = \int \frac{d^D l_1 \dots d^D l_L}{D_1^{n_1} D_2^{n_2} \dots D_N^{n_N}} \tag{2.2}$$

where the scalar functions D_α are linear polynomials with respect to s_{ij} . The functions D_α are assumed to be linearly independent and to form a complete basis in the sense that any non-zero linear combination of them depends on the loop momenta, and any s_{ik} can be expressed in terms of D_α . Thus, each integral is associated with a point in \mathbb{Z}^N . Some of the functions D_α correspond to the denominators of the propagators, the other correspond to the irreducible numerators. E.g., the K -legged L -loop diagram corresponds to $E = K - 1$ and the maximal number of denominators is $M = E + 3L - 2$, so that the rest $N - M = (L - 1)(L + 2E - 4)/2$ functions correspond to irreducible numerators. For vacuum diagrams, $M = 3(L - 1)$, and $N - M = (L - 2)(L - 3)/2$.

The IBP identities are based on the fact that, in the dimensional regularization, the integral of the total derivative is zero. They are derived from the identity

$$0 = \int d^D l_1 \dots d^D l_L O_{ikj}(\mathbf{n}) = \int d^D l_1 \dots d^D l_L \frac{\partial}{\partial l_i} \cdot q_{kj}(\mathbf{n}). \tag{2.3}$$

Performing the differentiation in the right-hand side and expressing the scalar products via D_α , we obtain the recurrence relation for the function J .

There is also another class of identities, called Lorentz-invariance (LI) identities due to the fact that the integral (2.2) is Lorentz scalar [6]. They have the form

$$p_i^\mu p_j^\nu \left(\sum_k p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}} \right) J(n_1, n_2, \dots, n_N) = 0 \quad (2.4)$$

The differential operator in braces is nothing but the generator of the Lorentz transformation in the linear space of scalar functions depending on p_k . If we explicitly act by the differential operator on the integrand, we obtain LI identity. Though these identities can be convenient in some cases, they can be easily represented as some linear combination of the IBP identities (see appendix) and will not be considered in the following.

For the reduction procedure to work, it is necessary to define some suitable ordering of the integrals, i.e., the ordering in \mathbb{Z}^N . First, one introduces the notion of sectors in \mathbb{Z}^N . The $(\theta_1, \dots, \theta_N)$ sector, where $\theta_i = 0, 1$, is a set of all points (n_1, \dots, n_N) in \mathbb{Z}^N whose coordinates obey the condition

$$\text{sign}(n_\alpha - 1/2) = 2\theta_\alpha - 1. \quad (2.5)$$

In particular, the point $(\theta_1, \dots, \theta_N)$ belongs to the $(\theta_1, \dots, \theta_N)$ sector, and can be referred to as the *corner point of the sector*. Owing to this definition, the integrals of the same sector have the same number of denominators. It is natural to consider the integrals with less denominators to be simpler. When the number of denominators coincides, we will consider the integrals with smaller total power of the numerators and denominators to be simpler. Then goes the number of the numerators and the last is the lexicographical ordering. Thus, two points $\mathbf{n} = (n_1, n_2, \dots)$ and $\mathbf{n}' = (n'_1, n'_2, \dots)$, are said to be ordered as $\mathbf{n} \prec \mathbf{n}'$ iff there exists $i_0, -2 < i_0 \leq N$, such that $n_{i_0} < n'_{i_0}$ and for any $i, -2 \leq i < i_0$ holds $n_i = n'_i$. Here n_{-2}, n_{-1} , and n_0 are determined as

$$n_{-2} = \sum_{\alpha=1}^N \Theta(n_\alpha - 1/2) = \sum_{\alpha=1}^N \theta_\alpha, \quad n_{-1} = \sum_{\alpha=1}^N |n_\alpha|, \quad n_0 = \sum_{\alpha=1}^N \Theta(-n_\alpha + 1/2). \quad (2.6)$$

The integral $J(\mathbf{n})$ is considered to be simpler than $J(\mathbf{n}')$ if $\mathbf{n} \prec \mathbf{n}'$. According to this ordering, the integral $J(\theta_1, \dots, \theta_N)$ is the simplest integral of $(\theta_1, \dots, \theta_N)$ sector.

3. Operator representation

Let us introduce, similar to ref. [4], the operators A_α and B_α acting on functions in \mathbb{Z}^N as follows

$$\begin{aligned} (A_\alpha f)(n_1, \dots, n_N) &= n_\alpha f(n_1, \dots, n_\alpha + 1, \dots, n_N), \\ (B_\alpha f)(n_1, \dots, n_N) &= f(n_1, \dots, n_\alpha - 1, \dots, n_N). \end{aligned} \quad (3.1)$$

Note that these operators act on function, but not on its arguments, and should not be confused with the conventional \mathbf{n}^\pm index shifting operators. Using these operators, we can

express the IBP identities as constraints on the function J having the form

$$\begin{aligned}
 -PJ &= 0, \\
 P &= a^{\alpha\beta} A_\alpha B_\beta + b^\alpha A_\alpha + c,
 \end{aligned}$$

where $a^{\alpha\beta}$, b^α , c are some coefficients. We will denote the operator, corresponding to the O_{ik} as P_{ik} :

$$-(P_{ik}J)(\mathbf{n}) = \int d^{\mathcal{D}}l_1 \dots d^{\mathcal{D}}l_L O_{ikj}(\mathbf{n}). \tag{3.2}$$

Note that the operators A_α, B_α form Weyl algebra,

$$[A_\alpha, B_\beta] = \delta_{\alpha\beta}. \tag{3.3}$$

Let \mathcal{L} be the left ideal generated by operators P_{ik} , i.e. a set, consisting of all operators, which can be represented as

$$\sum_{i,k} C_{ik} P_{ik}, \tag{3.4}$$

where C_{ik} are some polynomials of $A_1, \dots, A_N, B_1, \dots, B_N$. This ideal has a simple meaning: for any $L \in \mathcal{L}$ the relation

$$(LJ)(n_1, \dots, n_N) = 0 \tag{3.5}$$

is a linear combination of some IBP identities. In fact, any linear combination of the IBP identities can be represented in a more specific form

$$(LJ)(1, \dots, 1) = 0, \tag{3.6}$$

since shifting of the indices can be done by acting from the left with some powers of A_α or B_α . At first glance, the problem of reduction is equivalent to that of division with the remainder by the ideal \mathcal{L} , which is effectively solved by the construction of the Gröbner basis. However, there is an additional obstacle. Note that for any function f of N integer variables the following relation holds

$$(B_\alpha A_\alpha f)(1, \dots, 1) = 0. \text{ (no summation)} \tag{3.7}$$

Indeed,

$$(B_\alpha A_\alpha f)(1, \dots, 1) = (A_\alpha f)\left(1, \dots, \overset{\alpha}{0}, \dots, 1\right) = 0 \times f\left(1, \dots, \overset{\alpha}{1}, \dots, 1\right) = 0, \tag{3.8}$$

where the overscript α denotes the position of the index. Let \mathcal{R} be the right ideal generated by the elements $(B_1 A_1), \dots, (B_N A_N)$. By definition, it consists of all operators of the form

$$R = \sum_{\alpha} B_\alpha A_\alpha C_\alpha, \tag{3.9}$$

where C_α are some polynomials of $A_1, \dots, A_N, B_1, \dots, B_N$. It follows from eq. (3.7) that

$$(Rf)(1, \dots, 1) = 0. \tag{3.10}$$

Thus, for the reduction procedure to work, we have to have an algorithm of division with the remainder by the direct sum of the left ideal \mathcal{L} and the right ideal \mathcal{R} . That means that we have to invent the algorithm allowing the decomposition

$$p = L + R + r, \tag{3.11}$$

where $L \in \mathcal{L}$, $R \in \mathcal{R}$, and r is the simplest possible with respect to the ordering chosen. Even though the problem is clearly formulated, such algorithm appears to be unknown so far.

4. Lie-algebraic structure of the IBP identities

The operators

$$O_{ik} = \frac{\partial}{\partial l_i} \cdot q_k \tag{4.1}$$

form a closed algebra with the commutation relations

$$[O_{ik}, O_{jl}] = \delta_{il} O_{jk} - \delta_{jk} O_{il}. \tag{4.2}$$

We can easily check that the operators P_{ik} obey the same commutation relations as O_{ik} . This algebra is nothing but the algebra of the group of linear changes of variables

$$l_i \rightarrow M_{ik} q_k. \tag{4.3}$$

The operator O_{ik} corresponds to the infinitesimal transformation $l_i \rightarrow l'_i = l_i + \epsilon q_k$ in the sense that

$$f(s'_{lm}) d^{\mathcal{D}} l'_1 \dots d^{\mathcal{D}} l'_L = \left\{ f(s_{lm}) + \epsilon \left[\frac{\partial}{\partial l_i} \cdot q_k f(s_{lm}) \right] \right\} d^{\mathcal{D}} l_1 \dots d^{\mathcal{D}} l_L + O(\epsilon^2). \tag{4.4}$$

Note that the so-called symmetry relations are also the consequences of the invariance of the integrand under the action of some elements of this group.

The scaleless integral can be defined as the one which gains additional non-unity factor under some transformation (4.3). This definition corresponds to the conventional notion of scaleless integrals. E.g., owing to this definition, the integral

$$I = \int \frac{d^{\mathcal{D}} l_1 d^{\mathcal{D}} l_2}{(l_1 - p)^2 l_2^2 (l_1 - p - l_2)^2} \tag{4.5}$$

is scaleless as, under the transformation

$$\begin{aligned} l_1 &\rightarrow \alpha l_1 + (1 - \alpha) p, \\ l_2 &\rightarrow \alpha l_2, \end{aligned} \tag{4.6}$$

it transforms as

$$I \rightarrow \alpha^{2\mathcal{D}-6} I. \tag{4.7}$$

In dimensional regularization, the scaleless integrals are zero, as well as the integrals which differ from the scaleless ones by additional polynomial factor in the numerator. Thus, once the integral in the corner point of the sector is scaleless, the whole sector is zero. Let us prove a simple criterion of zero sectors.

Criterion 1. *If the solution of all IBP relations in the corner point of the sector $(\theta_1, \dots, \theta_N)$ results in the identity*

$$J(\theta_1, \dots, \theta_N) = 0, \tag{4.8}$$

then this sector is zero, i.e., all integrals of this sector are zero.

Indeed, by the condition, $j(\theta_1, \dots, \theta_N)$ can be represented as the action of some linear combination of O_{ik} on $j(\theta_1, \dots, \theta_N)$. Since these operators are generators of the transformation (4.3), we can conclude that $J(\theta_1, \dots, \theta_N)$ is scaleless and thus, the whole sector $(\theta_1, \dots, \theta_N)$ is zero. This criterion gives a simple and convenient way to determine zero sectors.

5. Excessiveness of the IBP identities set

In this section we describe some consequences of the algebraic structure of the IBP identities.

Proposition 1. *Let $L \geq 2$. Then of all $L(L + E)$ IBP identities we can consider only identities, generated by the operators:*

$$\begin{aligned} \frac{\partial}{\partial l_i} \cdot l_{i+1}, \quad i = 1, \dots, L, \quad l_{L+1} \equiv l_1 \\ \frac{\partial}{\partial l_1} \cdot p_j, \quad j = 1, \dots, E \\ \sum_{i=1}^L \frac{\partial}{\partial l_i} \cdot l_i \end{aligned} \tag{5.1}$$

Indeed, this set of operators form the multiplicative basis of the Lie-algebra (4.2), i.e., the rest of the operators can be obtained from the commutators of the chosen ones. The total number of the operators in the set (5.1) is $L + E + 1$, which is smaller than the original $L(L + E)$ for $L \geq 2$. This simple fact can be used for the construction of the reduction rules and also for the Laporta algorithm. Nevertheless, such system of the IBP identities is still overdetermined. In the asymptotics only $1/(L + E + 1)$ part of the identities gives new information.

Now we prove a more refined criterion for the identities which can be thrown away without loss of information. Let $S = \{P_1, P_2, \dots, P_K\}$ be some set of the IBP operators, and P be some IBP operator with the following property: its commutator with any $P_k \in S$ is a linear combination of $P_i \in S$. Then we have the following

Criterion 2. *If for some point $\mathbf{n} \in \mathbb{Z}^N$ the integral $J(\mathbf{n})$ can be expressed via simpler integrals with the help of the identities obtained from the operators in S , then the identity $(PJ)(\mathbf{n}) = 0$ can be represented as a linear combination of the identities obtained from the operators in S and the identities of the form $(PJ)(\mathbf{n}') = 0$ with $\mathbf{n}' \prec \mathbf{n}$.*

Proof. To prove it, we note that, since the integral $J(\mathbf{n})$ can be expressed via simpler integrals with the help of the identities obtained from the operators in S , there exist such polynomials C_l that for any function f

$$\sum_l (C_l P_l f)(\mathbf{1}) = f(\mathbf{n}) + o(\mathbf{n}), \tag{5.2}$$

$o(\mathbf{n})$ denotes here the linear combination of $f(\mathbf{n}')$ with $\mathbf{n}' \prec \mathbf{n}$. Now we substitute $f = PJ$:

$$\sum_l (C_l P_l PJ)(\mathbf{1}) = PJ(\mathbf{n}) + Po(\mathbf{n}), \tag{5.3}$$

and use the commutation relation

$$P_l P = P P_l + \sum_m c_{lm} P_m \tag{5.4}$$

where c_{lm} are some constants. We obtain

$$PJ(\mathbf{n}) = Po(\mathbf{n}) + \sum_l (C_l P P_l J)(\mathbf{1}) + \sum_{lm} (C_l c_{lm} P_m J)(\mathbf{1}) \tag{5.5}$$

Here $Po(\mathbf{n})$ denotes the linear combination of $PJ(\mathbf{n}')$ with $\mathbf{n}' \prec \mathbf{n}$, and the last two terms is a linear combination of the identities obtained from the operators in S . \square

Let us consider the sequence of the operators

$$\{\mathcal{P}_1, \dots, \mathcal{P}_N\} = \{P_{1,L+E}, \dots, P_{1,1}, P_{2,L+E}, \dots, P_{2,2}, \dots, P_{L,L+E}, \dots, P_{L,L}\} \tag{5.6}$$

Note that the number of the operators in this sequence equals to $N = L(L+1)/2 + LE$, the total number of the denominators and numerators in basis. This sequence has the following property: for any \mathcal{P}_i the criterion 3 applies with $S = \{\mathcal{P}_1, \dots, \mathcal{P}_{i-1}\}$. Suppose we have solved the identities, generated by operators $\{\mathcal{P}_1, \dots, \mathcal{P}_{i-1}\}$, then the identities, generated by \mathcal{P}_i should be solved only in points for which there are no reduction rules yet. On each step of this procedure the “dimension” of the set of such points is decreased by one, thus, when we have considered all operators in the sequence, we have only finite number of the integrals, which are not yet reduced. The rest $L(L-1)/2$ IBP identities can be used for the reduction of integrals in these points. Note, that the mean number of the identities, considered in each point is 1, as it should be. The choice of the sequence (5.6) is, of course, not unique.

Now we derive the criterion which can be used for the algorithm combining the “division with the remainder” and Laporta method.

Criterion 3. *Let in some sector the identities generated by some operator $P = \sum c_{ik} P_{ik}$ have the form*

$$(PJ)(\mathbf{n}) = J(\tilde{\mathbf{n}}) + o(\tilde{\mathbf{n}}) \tag{5.7}$$

and for any \mathbf{n} in the sector holds

$$\mathbf{n} \prec \tilde{\mathbf{n}}. \tag{5.8}$$

Then the identity generated by another operator $P' = \sum c'_{ik} P_{ik} \neq P$ in point $\tilde{\mathbf{n}}$, corresponding to the integral expressed by P , can be represented as a linear combination of the identities of the form PJ and the identities in simpler points.

Proof. To prove it, let us consider the identity

$$(PP'J)(\mathbf{n}) = (P'J)(\tilde{\mathbf{n}}) + (P'o)(\tilde{\mathbf{n}}) = P'PJ(\mathbf{n}) + ([P, P']J)(\mathbf{n}). \quad (5.9)$$

Thus

$$(P'J)(\tilde{\mathbf{n}}) = (P'PJ)(\mathbf{n}) - (P'o)(\tilde{\mathbf{n}}) + ([P, P']J)(\mathbf{n}). \quad (5.10)$$

The first term in the left-hand side is some linear combination of the IBP identities generated by P , and the last two terms contain only identities in the points $\mathbf{n}' \prec \tilde{\mathbf{n}}$. Thus, the identity $P'J(\tilde{\mathbf{n}})$ is dependent on the identities of the form PJ and on the identities in simpler points. \square

The basic idea of application of this criterion is the following. Consider the IBP identities in general point of the sector. We can either use the set of identities, generated by P_{ik} or by other $L(L+E)$ independent linear combinations of P_{ik} . In fact, it is natural to pass to such linear combinations, in which all most complex integrals are different ("solve" the identities in general point). Among these identities, select one, having the form (5.7) with $\mathbf{n} \prec \tilde{\mathbf{n}}$ and $\tilde{\mathbf{n}}$ corresponding to the integral not yet expressed. Determine the points of the sector corresponding to the integrals for which this identity does not work. Consider the other identities only in these points. Repeat the same steps, starting from the selection of the identity. After some iterations, we might be unable to find the identity matching the conditions. At this stage, we have to solve the rest of the identities only in the points, corresponding to the integrals, which are not yet expressed. E.g., we can use the Laporta method. The advantage of this combined approach is that the "dimension" of the set of points in which we use the Laporta method is usually essentially less than N .

6. Conclusion

In the present paper, we have considered the dependencies between the IBP identities. Account of these dependencies dramatically decreases the number of the identities to be considered. They come from the fact that the corresponding operators P_{ik} form a closed Lie algebra of the group of linear change of variables. Using this interpretation of the IBP identities, we have proved the simple criterion of the zero sectors. The two criteria of the excessiveness of an identity were proven. Probably, using the Criterion 2, it is possible to prove the finiteness of the number of master integrals in general case. Indeed, selecting the sequence of the operators as it was described, we decrease on each step the "dimension" of the set of the unexpressed integrals by one. Since the number of the operators in this sequence equals N , we are left with the set of "dimension" zero, i.e., consisting of finite number of points. The Criterion 3 can be used for the combined algorithms in which the Laporta reduction is performed on some subset of the points of the given sector with the dimension less than N . It was also shown that the Lorentz-invariance identities can be completely discarded.

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A. Expressing LI identities via IBP identities

In this appendix we will show that the LI identities always can be represented as a linear combination of the IBP identities.

Let us note that the integrand j in eq. (2.2) is a scalar function of q_i . Thus, the operator

$$\sum_{k=1}^{L+E} q_{k[\nu} \frac{\partial}{\partial q_k^{\mu]} = \sum_{k=1}^L l_{k[\nu} \frac{\partial}{\partial l_k^{\mu]} + \sum_{k=1}^E p_{k[\nu} \frac{\partial}{\partial p_k^{\mu]}, \quad (\text{A.1})$$

when acting on the integrand j , annihilates it identically. Indeed, this operator is nothing, but the generator of Lorentz transformations in the linear space of functions of q_i . Thus, the operator in eq. (2.4), when acting on the integrand, can be represented as follows:

$$\begin{aligned} p_i^\mu p_j^\nu \sum_{k=1}^E p_{k[\mu} \frac{\partial}{\partial p_k^{\nu]} j &= p_i^\mu p_j^\nu \sum_{k=1}^L l_{k[\nu} \frac{\partial}{\partial l_k^{\mu]} j - p_i^\mu p_j^\nu \sum_{k=1}^{L+E} q_{k[\nu} \frac{\partial}{\partial q_k^{\mu]} j \\ &= p_i^\mu p_j^\nu \sum_{k=1}^L l_{k[\nu} \frac{\partial}{\partial l_k^{\mu]} j \\ &= \sum_{k=1}^L \left[(p_i \cdot l_k) p_j \cdot \frac{\partial}{\partial l_k} - (p_j \cdot l_k) p_i \cdot \frac{\partial}{\partial l_k} \right] j \\ &= \sum_{k=1}^L \left[\frac{\partial}{\partial l_k} \cdot p_j (p_i \cdot l_k) - \frac{\partial}{\partial l_k} \cdot p_i (p_j \cdot l_k) \right] j \end{aligned}$$

Taking into account that the scalar products $(p_{i,j} \cdot l_k)$ in the last line can be expressed via D_α , we conclude, that any LI identity can be represented as a linear combination of the IBP identities.

References

- [1] K.G. Chetyrkin, A.L. Kataev and F.T. Tkachev, *New approach to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x space technique*, *Nucl. Phys. B* **174** (1980) 345.
- [2] K.G. Chetyrkin and F.T. Tkachev, *Integration by parts: the algorithm to calculate β -functions in 4 loops*, *Nucl. Phys. B* **192** (1981) 159.
- [3] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys. A* **15** (2000) 5087.
- [4] A.V. Smirnov and V.A. Smirnov, *Applying Gröbner bases to solve reduction problems for feynman integrals*, *JHEP* **01** (2006) 001.
- [5] A.V. Smirnov and V.A. Smirnov, *On the reduction of Feynman integrals to master integrals*, [arXiv:0707.3993](https://arxiv.org/abs/0707.3993).
- [6] T. Gehrmann and E. Remiddi, *Differential equations for two-loop four-point functions*, *Nucl. Phys. B* **580** (2000) 485 [[hep-ph/9912329](https://arxiv.org/abs/hep-ph/9912329)].